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Boundary controllability of Maxwell's equations with nonzero conductivity inside a cube, I: Spectral controllability

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Abstract

This is a first paper in a series of two. In both papers, we consider the question of control of Maxwell's equations in a homogeneous medium with positive conductivity by means of boundary surface currents. The domain under consideration is a cube, where the conductivity is allowed to take on any nonnegative value. An additional restriction imposed in order to make this problem more suitable for practical implementations is that the controls are applied over only one face of the cube. In this paper, the method of moments is employed to establish spectral controllability for the above case (meaning that any finite combination of eigenfunctions is controllable). In the companion paper [S.S. Krigman, C.E. Wayne, Boundary controllability of Maxwell's equations with nonzero conductivity inside a cube, II: Lack of exact controllability and controllability for very smooth solutions, J. Math. Anal. Appl. (2006), doi:10.1016/j.jmaa2006.02.102] it will be established, by modifying the calculations in [H.O. Fattorini, Estimates for sequences biorthogonal to certain complex exponentials and boundary control of the wave equation, in: New Trends in Systems Analysis, Proceedings of the International Symposium, Versailles, 1976, in: Lecture Notes in Control and Inform. Sci., vol. 2, Springer, Berlin, 1977, pp. 111–124], that exact controllability fails for this geometry regardless of the size of the conductivity term. However, we will also establish in [S.S. Krigman, C.E. Wayne, Boundary controllability of Maxwell's equations with nonzero conductivity inside a cube, II: Lack of exact controllability and controllability for very smooth solutions, J. Math. Anal. Appl. (2006),

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doi:10.1016/j.jmaa2006.02.102] controllability of solutions that are smooth enough that the Fourier coefficients of their initial data decay at a suitable exponential rate.

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1. Introduction

Suppose that Ω is a cube which consists of a medium which has constant values of electric permittivity, magnetic permeability, and conductivity. We will place no restrictions on the value of conductivity, other than that it must be nonnegative. Further, we will assume that the boundaries of Ω are made up of a perfectly conducting material, which acts as an impenetrable wall for the electro-magnetic waves. Both, in this paper and in the sequel [15], we will study the following controllability problem. Given some initial electric and magnetic fields in such a cube, is it possible to "steer" those fields to the desired state by means of control surface currents which are applied over only one face of the cube?

The controllability problem has an application to nuclear fusion, as was mentioned by Russell [25], as well as to the design of wave guides and cavities. Furthermore, the methods used in this problem could provide the framework for understanding acoustical problems like the localization or extinction of underwater sources (some of this has been discussed in [3,5]), as well as control of deflections in complex structures as was alluded to, e.g., in [26]. Further, an important application of exact controllability results, as was pointed out by Yamamoto [29], Nicaise [22] is the inverse source reconstruction problem.

Exact boundary controllability problems for Maxwell's equations have been addressed in the recent literature for the nonconductive medium. In that case, the authors were dealing with a system where the energy was either conserved [13,14,17,22,24,25] or dissipated through the boundary only [12]. In our case, the nonzero conductivity term implies the presence of free conducting electrons throughout the domain Ω. This has an effect of damping the energy throughout Ω with time. Additionally these conducting electrons are responsible for inducing a nonzero internal charge density throughout the region (unless the ratio of the electric permittivity to the conductivity is a constant, as for example happens in the homogeneous medium). These factors significantly complicate the mathematical description of the problem because the case without conductivity generates a self-adjoint Maxwell operator, while the present case generates a nonself-adjoint operator.

A frequently used approach to the problem of controllability of Maxwell's equation, is the Hilbert uniqueness method (HUM) of Lions [19,20]. This method was employed to study a variety of control problems in general star shaped regions, e.g., [14,17,18,22,24]. In HUM, the analysis is performed on the adjoint homogeneous system to the system one wants to control. The exact controllability problem is shown to be equivalent to the uniqueness of solutions of the adjoint system, together with establishing certain energy (or observability) inequalities. In [16] we attempted to prove the boundary controllability in a star shaped domain using the HUM. It turned out that even in the case of the homogeneous medium, we were still restricted in our ability to establish these inequalities except in the case of very small dissipation. A similar restriction was encountered by Triggiani in [28] when studying boundary controllability of a damped wave equation with the damping term being a bounded nonnegative operator on $L^2(\Omega)$. We wish to avoid such a restriction by considering a special domain and using a method more suitable for that domain—the method of moments (MOM).

On the one hand, the shape of a cube is simple enough so that, owing to symmetries, the eigenfunctions and eigenvalues of the Maxwell operator can be computed explicitly. On the other hand, this simplicity is deceptive. As was observed in [8], based on an earlier work of Russell [27], the sufficient geometric condition for exact boundary controllability of undamped scalar wave equation is satisfied after an arbitrarily small deformation of the cube Ω, when controls are applied over one face of Ω. Moreover, the controls applied over one face of a cube will violate the necessary geometric condition for the exact controllability of a scalar wave equation without internal dissipation, as given in [2] but small perturbations will also satisfy the sufficient conditions of [2].

We need to remark that we choose the cube over a parallelepiped only to simplify the calculations. In particular this will make the expressions for the eigenvalues a little less complicated and it will simplify the unwieldy normalization constants for the eigenfunctions. Otherwise, all of the analysis below will carry over to parallelepipedal domains as well.

Some of the earlier works on Maxwell's equations have considered a control problem on very special regions, made up of a homogeneous medium, where one can make use of symmetries (e.g., a sphere [13] or a cylinder [25]). Control problems on such regions were solved via a method of moments technique. With this method we are able to find explicitly the eigenfunctions of the Maxwell operator and solve for the controls explicitly. It is not applicable to a general region, but it will be our method of choice here.

Because Maxwell's equations in homogeneous media can be re-written as vector wave equations (with dissipation in our case) we were able to adapt the techniques used in the controllability of the scalar wave equation. Many works of Russell and various collaborators were crucial in the development of this topic. The work [26] treats the controllability of vibrations of a string with controls being applied throughout the string. This work provides many valuable tools of "nonharmonic Fourier series"—a study of the properties of sets of complex exponentials $\{e^{i\lambda_n t}\}$ in $L^p(-T,T)$. Together with [11] it gives us the tools to determine whether the sequences of this type, which will be encountered in this work, are minimal (hence possess biorthogonal sequences). Boundary controllability of the wave equation in a three-dimensional sphere is studied in [10]. Many useful techniques on setting up the systems of equations for the method of moments in a way which yields a natural solution to the control problem in terms of sequences biorthogonal to certain complex exponentials, were borrowed from this work.

Fattorini [8] has studied the problem in the case of boundary controllability of a scalar wave equation with controls being applied over one face of a parallelepiped. He has found that given any $T > 0$, it is not possible to steer any Sobolev space of initial data to zero via controls in $L^2(\Gamma \times (0,T))$ if Γ is only one face of the boundary. Fattorini has also found that in the case when the initial data is very smooth (roughly speaking, when the Fourier coefficients decrease faster than inverse of the exponent of the index), it can be steered to zero in the time required for the wave to travel across the domain and back. In the sequel [15], we have adapted Fattorini's techniques, while accounting for the differences between undamped scalar wave equation and damped Maxwell's equations, in order to establish similar results for our case.

We must mention that one of the key results established in [15], a lower bound on the norms of certain functions, does not entirely agree with a similar result derived in [8]. We believe that these difference will remain even in the case of the scalar wave equation, discussed in [8]. This implies that the space of the initial data for which the control problem is shown to not have a solution is smaller than is claimed in [8].

We now give a more explicit description of the problem we consider. Assume that Ω is a box of dimensions $\pi \times \pi \times \pi$. We are going to consider controls along the upper face of this cube. Thus

$$\Omega = [0, \pi] \times [0, \pi] \times [0, \pi], \tag{1}$$

$$\Gamma = \{(x, y, z) \in \Omega\colon z = \pi\} \tag{2}$$

and the equations for which we consider the boundary controllability problem are

$$\nabla \times E + \dot{H} = 0 \quad \text{in } \Omega,\ t > 0, \tag{3}$$

$$\dot{E} + \sigma E - \nabla \times H = 0 \quad \text{in } \Omega,\ t > 0, \tag{4}$$

$$\nabla \cdot H = 0 \quad \text{in } \Omega,\ t > 0, \tag{5}$$

$$E(0) = E^0, \quad H(0) = H^0 \quad \text{in } \Omega \tag{6}$$

subject to the boundary condition

$$n \times H = -J \quad \text{on } \Gamma,\ t > 0, \qquad n \times H = 0 \quad \text{on } \partial\Omega \backslash \Gamma,\ t > 0. \tag{7}$$

We wish to find controls $J(\underline{x}, t)$ such that $E(\underline{x}, T) = H(\underline{x}, T) = 0$ for some $T > 0$. Recall that because the problem is linear and because the initial value problem for Maxwell's equations is well-posed both in forward and backward time, the ability to drive an arbitrary initial data to a null state is equivalent to the ability to drive that data to any other arbitrary state.

In the remainder of the introductory section we will describe an outline of this work. Relevant function spaces and main results from both the current paper and the sequel [15] are presented in Section 2. First, we will need to obtain the formal solution to the control problem. In order to do that, we need to determine the eigenfunctions and the eigenvalues of the Maxwell operator, which is done in Section 3. Section 4 established the well-posedness of solutions of Maxwell's equations with boundary input. In Section 5, we represent the initial data (6) in terms of an infinite summation with respect to this basis. At the same time we will represent the boundary control function as an infinite summation of functions of time over the basis functions of $L^2(\Gamma)^3$. Direct substitutions of these expansions into the integral representation of the control problem will yield an equivalent moment problem consisting of a countably infinite system of equations. In Section 6, this system will be solved for the unknowns which are the coefficients of functions of time biorthogonal to certain complex exponentials, resulting in the controls which bring the initial data to rest. In Section 7, the existence of such biorthogonal sequences of functions will be discussed and we will quote some relevant results from nonharmonic Fourier series. It will then be shown that the controls which bring to rest any finite combination of eigenvectors in the initial data belong to the function space $L^2(0, T; L^2(\Gamma)^3)$, provided that T is large enough, establishing main result of the current paper.

It will also become clear that a bound needs to be obtained on the norms of these biorthogonal sequences in order to be able to tell whether the controls obtained in this way can be bounded in any norm, particularly the norm in $L^2(0, T; L^2(\Gamma)^3)$. In the companion paper [15], we will modify the analysis in [8] to show that the norm of each element of the biorthogonal sequence may be bounded from *below*. This result will in turn lead us to the conclusion that any Sobolev space will possess initial conditions such that the resulting electromagnetic field cannot be steered to zero by the controls residing in $L^2(0, T; L^2(\Gamma)^3)$ in any finite time $T > 0$ (because the initial conditions are not smooth enough).

We will then establish an *upper* bound on the norms of these biorthogonal functions by actually constructing them. This will allow us to quantify just how smooth the initial data must be in order for the controls to reside in the space $L^2(0, T; L^2(\Gamma)^3)$.

2. Function spaces and main results

We will list below some function spaces that we will be using along with their inner products. We choose notation which is consistent with [22]. Define first the following spaces:

$$\mathcal{H} = L^2(\Omega)^3 \times L^2(\Omega)^3.$$

Thus its inner product is given by

$$\left(\begin{pmatrix}\phi_1\\ \psi_1\end{pmatrix}, \begin{pmatrix}\phi_2\\ \psi_2\end{pmatrix}\right)_{\mathcal{H}} = \int_\Omega \{\phi_1 \cdot \bar{\phi}_2 + \psi_1 \cdot \bar{\psi}_2\}\, dx.$$

Also define two more spaces provided with the usual $L^2(\Omega)^3$ norm:

$$J(\Omega, 1) = \{\chi \in L^2(\Omega)^3\colon\ \nabla \cdot (\chi) = 0\},$$
$$\hat{J}(\Omega, 1) = \{\chi \in J(\Omega, 1)\colon\ \chi \cdot n|_{\partial\Omega} = 0\}.$$

With these spaces we can define some more:

$$H(\mathrm{curl}, \Omega) = \{\chi \in L^2(\Omega)^3\colon\ \nabla \times \chi \in L^2(\Omega)^3\},$$
$$H_0(\mathrm{curl}, \Omega) = \{\chi \in L^2(\Omega)^3\colon\ \nabla \times \chi \in L^2(\Omega)^3,\ \chi \times n|_{\partial\Omega} = 0\},$$
$$J_\tau^1(\Omega, 1) = \{\chi \in H_0(\mathrm{curl}, \Omega)\colon\ \nabla \cdot (\chi) = 0\},$$
$$J_\nu^1(\Omega, 1) = \hat{J}(\Omega, 1) \cap H(\mathrm{curl}, \Omega)$$

all four spaces provided with the norm

$$\|\chi\|^2_{H(\mathrm{curl},\Omega)} = \|\chi\|^2_{L^2(\Omega)^3} + \|\nabla \times \chi\|^2_{L^2(\Omega)^3}.$$

Also set

$$\mathcal{H}_1 = J(\Omega, 1) \times \hat{J}(\Omega, 1),$$
$$\mathcal{K} = J_\tau^1(\Omega, 1) \times J_\nu^1(\Omega, 1)$$

with the norm on $\mathcal{H}_1$ and $\mathcal{K}$ defined based on the norms on the their constituent spaces.

Remark 1. Because our chosen domain Ω, the cube, is convex, any function χ which belongs to either $J_\tau^1(\Omega, 1)$ or $J_\nu^1(\Omega, 1)$ must belong to $H^1(\Omega)^3$, e.g., [4].

We now turn our attention to three major theorems established in this paper.

Theorem 1. *Suppose Ω, Γ are as in* (1), (2) *and Ω is made up of homogeneous medium with any nonnegative conductivity value $\sigma \geqslant 0$. Then for $T > 2\pi$ the system* (3)–(7) *is spectrally controllable* (*meaning that any finite combination of eigenfunctions of the Maxwell operator may be steered to a null state*) *in time T. The controls*

$$J(x, t) \in L^2(0, T; L^2(\Gamma)^3).$$

In particular, the controls are applied over just one face of Ω.

As a byproduct of the proof, we obtain explicit formulas for controls corresponding to each mode. We also note that Γ may be any face of the cube, however, for concreteness, all the analysis will be performed for Γ as in (2). Our next theorem, to be established in the sequel [15], states that spectral controllability is the strongest result possible.

Theorem 2. *There exist functions* $\{H^0, E^0\} \in \mathcal{K}$*, such that the control problem has no solution* $J \in L^2(0, T; L^2(\Gamma)^3)$ *for any time* $T > 0$.

The proof will also reveal that the set of all controllable initial states is a countable union of nowhere dense subsets (a category one set) of $\mathcal{H}$, meaning that the set of the states which cannot be exactly controlled is a category two set.

The last major result, also to be established in [15], shows that for very smooth initial data we can solve the control problem. Represent the initial data (6) in terms of Fourier coefficients E^0_{lmnk}, H^0_{lmnk} as

$$E^0 = \sum_{lmn} \sum_{k=0,1,2} E^0_{lmnk} \Psi_{lmnk}(x), \qquad H^0 = \sum_{lmn} \sum_{k=0,1,2} H^0_{lmnk} \Phi_{lmnk}(\underline{x}) \tag{8}$$

with $\Psi_{lmnk}(x)$, $\Phi_{lmnk}(x)$ being the basis functions for $J^1_\nu(\Omega, 1)$ and $J^1_\tau(\Omega, 1)$, respectively, which will be defined in the next section. Define

$$\mu(l, m) \stackrel{\text{def}}{=} \sqrt{\left(l^2 + m^2\right) - (\sigma/2)^2}.$$

Theorem 3. *Let* $T > 2\pi$. *Let* E^0, H^0 *be given by* (8) *with*

$$\sum_{lmn} \sum_{k=1,2} \left(\frac{|E^0_{lmnk}|^2}{K^2(l,m,n,k)} + \frac{|H^0_{lmnk}|^2}{K^2(l,m,n,k)} \right) e^{2\pi\mu(l,m)} = \tau^2\left(E^0, H^0\right) < \infty$$

then the control problem has a solution $J(\underline{x}, t) \in L^2(0, T; L^2(\Gamma)^3)$ *such that*

$$\left\| J(\underline{x}, t) \right\|_{L^2(0,T;L^2(\Gamma)^3)} \leqslant C\tau\left(E^0, H^0\right).$$

The normalization constant $K(l, m, n, k)$ *is defined in* (16) *and* (17) *below.* C *is a constant independent of the initial data.*

3. Eigenfunctions for Maxwell operator in the homogeneous medium in a cube

Before proceeding further we need to define some key operators. We start with the Maxwell operator $\mathcal{A}$ defined as

$$\mathcal{A} = \begin{pmatrix} 0 & -\mu^{-1}\text{curl} \\ \epsilon^{-1}\text{curl} & -\epsilon^{-1}\sigma \end{pmatrix}.$$

Following [23], $\mathcal{A}$ can be written as $\mathcal{A} = \mathcal{A}_0 + \mathcal{B}$, where

$$\mathcal{A}_0 = \begin{pmatrix} 0 & -\mu^{-1}\text{curl} \\ \epsilon^{-1}\text{curl} & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{B} = \begin{pmatrix} 0 & 0 \\ 0 & \epsilon^{-1}\sigma \end{pmatrix}.$$

In this section we are going to list the eigenfunctions of the Maxwell operator for the geometry described above. It can easily be shown that in a homogeneous medium solutions with divergence free initial data remain divergence free. Therefore the domain of this operator is $J^1_\tau(\Omega,1)\times J^1_\nu(\Omega,1)$ (which is dense in $J(\Omega,1)\times\hat{J}(\Omega,1)$ [22]), as in the conservative case. In the case of a Maxwell's operator without dissipation, $\mathcal{A}_0$, Theorem 1 [6, p. 266] yields that $\mathcal{A}_0^* = -\mathcal{A}_0$. Therefore $i\mathcal{A}_0$ is a self-adjoint operator. The compact embedding of $D(\mathcal{A}) = D(\mathcal{A}_0) = D(i\mathcal{A}_0) \Subset J(\Omega,1)\times\hat{J}(\Omega,1)$, observed in [22] allows us to use the Spectral Theorem [31], to conclude that the eigenvectors of $\mathcal{A}_0$ form a complete orthonormal system for $J(\Omega,1)\times\hat{J}(\Omega,1)$. Consequently, the eigenvectors of $\mathcal{A}$ which are the same as the eigenvectors of $\mathcal{A}_0$ form a complete orthonormal system for $J(\Omega,1)\times\hat{J}(\Omega,1)$. Recall that both the electric and magnetic fields in the homogeneous medium without dissipation satisfy the three-dimensional vector wave equation. Therefore in order to find the eigenfunctions of the Maxwell operator, we will simply look for the eigenfunctions of the $-\Delta_{\underline{x}}$ operator in 3 dimensions which satisfies the divergence free conditions and also the appropriate boundary conditions: requiring that either the tangential or the normal component must vanish at the boundary.

3.1. The basis for $J^1_\nu(\Omega,1)$

Those eigenfunctions of $-\Delta_{\underline{x}}$ belonging to the space $J^1_\nu(\Omega,1)$, denoted by $\{\Psi_{lmnk}\}$, must satisfy $\Psi_{lmnk}\cdot n = 0$ on the boundary $\partial\Omega$. They may be written the form of the triplets $\Psi^{(i)}_{lmn}(x,y,z)$, $i=1,2,3$, $l,m,n\geqslant 1$

$$\Psi^{(1)}_{lmn}(x,y,z) = A\sin(lx)\cos(my)\cos(nz), \tag{9}$$

$$\Psi^{(2)}_{lmn}(x,y,z) = B\cos(lx)\sin(my)\cos(nz), \tag{10}$$

$$\Psi^{(3)}_{lmn}(x,y,z) = C\cos(lx)\cos(my)\sin(nz). \tag{11}$$

Two of the three coefficients A, B, and C are arbitrary. The third coefficient must be chosen in a such way in order to keep Ψ solenoidal.

The basis functions for $J^1_\nu(\Omega,1)$, normalized in the $L^2(\Omega)^3$ norm, form "pairs" of triplets listed below. One pair corresponds to $B=0$ and another one to $A=0$.

$$\Psi^{(1)}_{lmn1}(x,y,z) = \sqrt{\frac{8}{\pi^3(1+\frac{l^2}{n^2})}}\sin(lx)\cos(my)\cos(nz), \quad l,m,n\geqslant 1,$$

$$\Psi^{(2)}_{lmn1}(x,y,z) = 0, \quad l,m,n\geqslant 1,$$

$$\Psi^{(3)}_{lmn1}(x,y,z) = -\frac{l}{n}\sqrt{\frac{8}{\pi^3(1+\frac{l^2}{n^2})}}\cos(lx)\cos(my)\sin(nz), \quad l,m,n\geqslant 1,$$

and

$$\Psi^{(1)}_{lmn2}(x,y,z) = 0, \quad l,m,n\geqslant 1,$$

$$\Psi^{(2)}_{lmn2}(x,y,z) = \sqrt{\frac{8}{\pi^3(1+\frac{m^2}{n^2})}}\cos(lx)\sin(my)\cos(nz), \quad l,m,n\geqslant 1,$$

$$\Psi^{(3)}_{lmn2}(x,y,z) = -\frac{m}{n}\sqrt{\frac{8}{\pi^3(1+\frac{m^2}{n^2})}}\cos(lx)\cos(my)\sin(nz), \quad l,m,n\geqslant 1.$$

For clarity we list the normalized functions which correspond to one of the indices l,m,n being zero separately:

$$\Psi^{(1)}_{0mn0}(x,y,z)=0, \quad m,n\geqslant 1,$$

$$\Psi^{(2)}_{0mn0}(x,y,z)=\sqrt{\frac{4n^2}{\pi^3(n^2+m^2)}}\sin(my)\cos(nz), \quad m,n\geqslant 1,$$

$$\Psi^{(3)}_{0mn0}(x,y,z)=-\sqrt{\frac{4m^2}{\pi^3(n^2+m^2)}}\cos(my)\sin(nz), \quad m,n\geqslant 1,$$

$$\Psi^{(1)}_{l0n0}(x,y,z)=-\sqrt{\frac{4n^2}{\pi^3(n^2+l^2)}}\sin(lx)\cos(nz), \quad l,n\geqslant 1,$$

$$\Psi^{(2)}_{l0n0}(x,y,z)=0, \quad l,n\geqslant 1,$$

$$\Psi^{(3)}_{l0n0}(x,y,z)=\sqrt{\frac{4l^2}{\pi^3(n^2+l^2)}}\sin(nz)\cos(lx), \quad l,n\geqslant 1,$$

$$\Psi^{(1)}_{lm00}(x,y,z)=\sqrt{\frac{4m^2}{\pi^3(l^2+m^2)}}\sin(lx)\cos(my), \quad l,m\geqslant 1,$$

$$\Psi^{(2)}_{lm00}(x,y,z)=-\sqrt{\frac{4l^2}{\pi^3(l^2+m^2)}}\cos(lx)\sin(my), \quad l,m\geqslant 1,$$

$$\Psi^{(3)}_{lm00}(x,y,z)=0, \quad l,m\geqslant 1.$$

Note on notation. When $l>0$, $m>0$, and $n>0$, index k can be either 1 or 2. When only $l=0$, or $m=0$, or $n=0$ (the case of two or more indices from the set $\{l,m,n\}$ being zero will be addressed below), we no longer need the fourth index, k. However, we retain it and set $k=0$ in order to keep our notation consistent.

3.2. The basis for $J^1_\tau(\Omega,1)$

Eigenfunctions of $-\Delta_{\underline{x}}$ belonging to $J^1_\tau(\Omega,1)$, will be denoted by $\{\Phi_{lmnk}\}$. These eigenfunctions must satisfy $\Phi_{lmnk}\times n=0$ on $\partial\Omega$. From the form of the operator $\mathcal{A}_0$, we see that taking the curl of $\Psi_{lmnk}(x,y,z)$ yields the eigenfunctions, $\Phi^{(i)}_{lmnk}(x,y,z)$, $i=1,2,3$, $k=1,2$, $l,m,n\geqslant 1$. We will not need their explicit formulation in this work. This can be found in [16]. Here, we just mention the relationship between $\Phi_{lmnk}(x,y,z)$ and $\Psi_{lmnk}(x,y,z)$

$$\alpha_{lmn}\Psi_{lmnk}=\nabla\times\Phi_{lmnk}, \quad \alpha_{lmn}\Phi_{lmnk}=\nabla\times\Psi_{lmnk} \quad \forall k,l,m,n. \tag{12}$$

Where we defined

$$\alpha_{lmn}=\sqrt{\left(l^2+m^2+n^2\right)}. \tag{13}$$

3.3. Justification of completeness

In this paragraph we explain in somewhat more detail why the eigenfunctions constructed above form a complete set. Recall that as mentioned in the introduction of this section the eigenvectors of the dissipative operator $\mathcal{A}$ coincide with those of the dissipationless operator $\mathcal{A}_0$.

Furthermore, each component of the solution in the dissipationless case satisfies the three-dimensional wave equation and thus the components of the eigenfunctions of $\mathcal{A}_0$ can be written in the form

$$\big(A_1\sin(lx)+B_1\cos(lx)\big)\big(A_2\sin(my)+B_2\cos(my)\big)\big(A_3\sin(nz)+B_3\cos(nz)\big).$$

If we now use the fact that $\Phi_{lmnk}=\nabla\times\Psi_{lmnk}$, and impose the boundary conditions that Ψ_{lmnk} be tangential to the boundary and Φ_{lmnk} be normal to the boundary, we find that the only allowable combinations of coefficients in the eigenfunctions Ψ_{lmnk} are those given in (9)–(11). Moreover, since $J^1_\nu(\Omega,1)\subset L^2(\Omega)^3$, and since the vectors in (9)–(11) obviously form a basis for $L^2(\Omega)^3$, we see that no other combination of sines or cosines needs to be included in the basis for $J^1_\nu(\Omega,1)$. Finally, imposing the condition that both the electric and magnetic field are solenoidal leads to the subset of basis elements included in Section 3.1 (see [16] for details).

3.4. The basis for $L^2(\Gamma)^3$ and its relationship to $\{\Psi_{lmnk}|_\Gamma\}$

Finally we need to consider one more set of functions. We will need to represent the controls, which will be derived with the aid of a solution to the adjoint problem defined below, in terms of the sum of some time dependent functions and a basis of $L^2(\Gamma)^3$. Our particular choice of a basis is guided by the requirement that it must closely resemble the functions $\{\Psi_{lmnk}\}$ restricted to Γ, which will facilitate the moment problem calculations.

Remark 2. The problem is three-dimensional. Therefore, we must consider the face Γ, which is a square, as an object in three-dimensional space R^3.

Define an orthonormal set $\{\bar\Psi_{lmk}(x,y)\}$, $k=0,1,2$, $l,m=0,1,2,\dots$, but not both $l=m=0$, as:

$$\bar\Psi^{(1)}_{lm1}(x,y)=\frac{2}{\pi}\sin(lx)\cos(my),\quad \bar\Psi^{(2)}_{lm1}(x,y)=0,\quad \Psi^{(3)}_{lm1}(x,y)=0,\quad l,m>0,$$

$$\bar\Psi^{(1)}_{lm2}(x,y)=0,\quad \bar\Psi^{(2)}_{lm2}(x,y)=\frac{2}{\pi}\cos(lx)\sin(my),\quad \bar\Psi^{(3)}_{lm2}(x,y)=0,\quad l,m>0.$$

Again, we write out the case where one of $l,m=0$ separately

$$\bar\Psi^{(1)}_{0m0}(x,y)=0,\qquad \bar\Psi^{(2)}_{0m0}(x,y)=\frac{\sqrt2}{\pi}\sin(my),\qquad \bar\Psi^{(3)}_{0m0}(x,y,z)=0,$$

$$\bar\Psi^{(1)}_{l00}(x,y)=\frac{\sqrt2}{\pi}\sin(lx),\qquad \bar\Psi^{(2)}_{l00}(x,y)=0,\qquad \bar\Psi^{(3)}_{l00}(x,y)=0.$$

The set $\{\bar\Psi_{lmk}\}$ forms an orthonormal basis of $L^2(\Gamma)^3$ and also relates to the basis functions $\{\Psi_{lmnk}\}$ of $J^1_\nu(\Omega,1)$ in the following way:

$$\int_\Gamma \bar\Psi_{l_1m_1k_1}(x,y)\cdot\Psi_{l_2m_200}=K(l_2,m_2,0,k_1)\delta_{l_1l_2}\delta_{m_1m_2}\quad \text{if } n=0,\ l_1*m_1\neq0, \tag{14}$$

$$\int_\Gamma \bar\Psi_{l_1m_1k_1}(x,y)\cdot\Psi_{l_2m_2n_2k_2}=K(l_2,m_2,n_2,k_2)\delta_{l_1l_2}\delta_{m_1m_2}\delta_{k_1k_2}\quad \text{otherwise}. \tag{15}$$

Remark 3. The two different formulas above serve to emphasize the anomalous case which corresponds to $n=0$. If $l\neq0$ and $m\neq0$, Ψ_{lm00} is not orthogonal to either $\bar\Psi_{lm1}$ or $\bar\Psi_{lm2}$.

The constant of integration, $K(l,m,n,k)$, is defined by

$$K(l,m,n,k) = \int_\Gamma \bar{\Psi}_{lmk} \cdot \Psi_{lmn0}\, d\Gamma \quad \text{if } n=0,\ l*m \neq 0, \tag{16}$$

$$K(l,m,n,k) = \int_\Gamma \bar{\Psi}_{lmk} \cdot \Psi_{lmnk}\, d\Gamma \quad \text{otherwise.} \tag{17}$$

The values of $K(l,m,n,k)$ can easily be calculated or found in [16]. We only note here that depending upon the values of indices l,m,n,k the constant

$$K(l,m,n,k) \sim \frac{1}{l} \quad \text{or} \quad K(l,m,n,k) \sim \frac{1}{m}. \tag{18}$$

Remark 4. When more than one index from the set $\{l,m,n\}$ is zero, we have $\Psi_{lmn0} = 0$, $\Phi_{lmn0} = 0$ and $K(l,m,n,0) = 0$. In particular, the operator $\mathcal{A}_0$ has no nonzero eigenfunctions or eigenvalues which correspond to the index $(l,0,0)$ or to $(0,m,0)$. This will become important in setting up the moment problem.

4. Well-posedness of solutions

In order to make precise the way in which solution to the system (3)–(7) are understood, we need to set up the (negative) adjoint system, given with respect to the $\mathcal{H}$ norm. This system takes the following form:

$$\dot{\phi} + \nabla \times \psi = 0 \quad \text{in } \Omega,\ t > 0, \tag{19}$$

$$\dot{\psi} - \nabla \times \phi - \sigma\psi = 0 \quad \text{in } \Omega,\ t > 0, \tag{20}$$

$$\nabla \cdot \phi = \nabla \cdot \psi = 0 \quad \text{in } \Omega,\ t > 0, \tag{21}$$

$$\phi(0) = \phi^0 \in J^1_\tau(\Omega, 1), \qquad \psi(0) = \psi^0 \in J^1_\nu(\Omega, 1). \tag{22}$$

Existence and uniqueness of strong solutions of (19)–(22) was established in [16, Chapter 4] by using denseness of $\mathcal{K}$ in $\mathcal{H}_1$ and applying Lumer–Phillips theorem. We will proceed as in [18,24]. Let $\{\phi,\psi\}$ be a solution of (19)–(22). Let us consider the system (3)–(7) with $\{H^0, E^0\} \in \mathcal{K}'$, where $\mathcal{K}'$ is dual of $\mathcal{K}$ with respect to $\mathcal{H}_1$ ($\mathcal{K} \subset \mathcal{K}'$) and let $J \in L^2(0,T; L^2(\Gamma)^3)$. Integration by parts leads to

$$\bigl\langle \bigl\{H(t), E(t)\bigr\}, \bigl\{\phi(t), \psi(t)\bigr\}\bigr\rangle = \bigl\langle \bigl\{H^0, E^0\bigr\}, \bigl\{\phi^0, \psi^0\bigr\}\bigr\rangle - \int_0^t \int_\Gamma \psi \cdot J\, d\Gamma\, ds, \tag{23}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product on $\mathcal{H}$.

The first bracket in (23) is by definition a linear functional on $\{\phi^0,\psi^0\}$. Since $\{\phi(t),\psi(t)\}$ remains in $\mathcal{K}$ for all t when the medium is homogeneous [16] then by existence and uniqueness of solutions to (19)–(22), the map $\{\phi^0,\psi^0\} \to \{\phi(t),\psi(t)\}$ is an isomorphism from $\mathcal{K} \to \mathcal{K}$. Thus the integral in the expression (23) is also a linear functional on the elements of $\mathcal{K}$. To show uniqueness of this solution, we bound (23) with the brackets reinterpreted as the duality pairings between $\mathcal{K}$ and $\mathcal{K}'$

$$\begin{aligned} &\bigl|\bigl\langle \bigl\{H(t), E(t)\bigr\}, \bigl\{\phi(t), \psi(t)\bigr\}\bigr\rangle_{\mathcal{K}',\mathcal{K}}\bigr| \\ &\quad \leqslant \bigl\|\bigl\{H^0, E^0\bigr\}\bigr\|_{\mathcal{K}'} \bigl\|\bigl\{\phi^0, \psi^0\bigr\}\bigr\|_{\mathcal{K}} + \|J\|_{L^2(0,t;L^2(\Gamma)^3)} \|\psi\|_{L^2(0,t;L^2(\Gamma)^3)}. \end{aligned} \tag{24}$$

Combining (24) with "direct" inequality established in [16] implies the existence of a constant C_2 which does not depend on any initial data such that

$$\begin{aligned}&\left|\left\langle\left\{H(t),E(t)\right\},\left\{\phi(t),\psi(t)\right\}\right\rangle_{\mathcal{K}'_\epsilon,\mathcal{K}_\epsilon}\right|\\&\quad\leqslant\left\|\left\{H^0,E^0\right\}\right\|_{\mathcal{K}'_\epsilon}\left\|\left\{\phi^0,\psi^0\right\}\right\|_{\mathcal{K}_\epsilon}+\|J\|_{L^2(0,t;L^2(\Gamma))}C_2\left\|\left\{\phi^0,\psi^0\right\}\right\|_{\mathcal{K}_\epsilon}.\end{aligned}$$

Consequently right side of (23) is a linear functional on $\mathcal{K}$ which defines the solution

$$\left\{H(t),E(t)\right\}\in C\left([0,t];\mathcal{K}'\right)$$

for (19)–(22) for any time $t\geqslant 0$.

Remark 5. Recently, the well-posedness of the system which generalizes our system (19)–(22) has also been established in [7, Section 3]. Nevertheless, the above analysis still serves the purpose of justifying the formula (23).

5. Reduction to a moment problem

We turn our attention to setting up the infinite system of equations for the moment problem. Note that we are able to solve the system (19)–(22) explicitly as an infinite series with respect to the eigenfunctions of Section 3. We are going to need these solutions to be obtained for enough initial data points so that we may set up a moment problem and for a variety of values of σ to account for different types of behavior of solutions of (19)–(22). Since our goal is to control the solutions to zero, at time $t=T$ we require $\{H(T),E(T)\}=0$, and therefore expression (23) becomes

$$\left\langle\left\{H^0,E^0\right\},\left\{\phi^0,\psi^0\right\}\right\rangle=\int_0^t\int_\Gamma\psi\cdot J\,d\Gamma\,ds.$$

For clarity we rewrite the above expression as

$$\int_\Omega H(\underline{x},0)\cdot\phi(\underline{x},0)\,d\underline{x}+\int_\Omega E(\underline{x},0)\cdot\psi(\underline{x},0)\,d\underline{x}=\int_0^t\int_\Gamma\psi\cdot J\,d\Gamma\,ds.\tag{25}$$

Denoting the triplet via the multi-index $\underline{n}\overset{\text{def}}{=}(l,m,n)$, we define

$$H^0_{\underline{n}k}=\int_\Omega H(\underline{x},0)\cdot\Phi_{\underline{n}k}(\underline{x})\,d\underline{x},\qquad E^0_{\underline{n}k}=\int_\Omega E(\underline{x},0)\cdot\Psi_{\underline{n}k}(\underline{x})\,d\underline{x}.\tag{26}$$

We will obtain the moment problem by plugging in various sets of the initial data together with the corresponding solutions of system (19)–(22), derived in [16], into (25). Combining them with the definition (26) is going to yield the needed pairs of equations for every multi-index $\underline{n}$. Since the orthonormal set $\{\bar{\Psi}_{lmk}\}$ forms a basis of $L^2(\Gamma)^3$ we may write

$$\begin{aligned}J(\underline{x},t)=e^{-\frac{\sigma}{2}t}\Bigg[&\sum_{m=1}^{\infty}\gamma_{0m0}(t)\bar{\Psi}_{0m0}(\underline{x})+\sum_{l=1}^{\infty}\gamma_{l00}(t)\bar{\Psi}_{l00}(\underline{x})\\&+\sum_{l,m=(1,1)}^{\infty}\sum_{k=1,2}\gamma_{lmk}(t)\bar{\Psi}_{lmk}(\underline{x})\Bigg].\end{aligned}\tag{27}$$

The solutions of the moment problem will result in the coefficients which will give an expression for each function $\gamma_{lmk}(t)$ making up (27) in terms of the functions biorthogonal to certain complex exponentials. Expression (25) shows that the only part of the solution of (19)–(22) that enters the moment calculation is $\psi(\underline{x},t)$. We can show that ψ satisfies the damped wave equation

$$\ddot{\psi} - \sigma\dot{\psi} - \Delta\psi = 0. \tag{28}$$

Different eigenmodes of Eq. (28) corresponding to the multi-index $\underline{n}$ will have qualitatively different behavior depending on whether the value

$$4\alpha_{lmn}^2 - \sigma^2 < 0, \quad 4\alpha_{lmn}^2 - \sigma^2 > 0, \quad \text{or} \quad 4\alpha_{lmn}^2 - \sigma^2 = 0.$$

We shall refer to these modes as "weakly dissipative," "strongly dissipative," or "intermediate," respectively.

5.1. Moment problem for weakly dissipative modes

Define the quantity

$$h_M(l,m,n) = \sqrt{4\left(l^2+m^2+n^2\right)-\sigma^2}, \quad \text{if } n \geqslant 0, \tag{29}$$

$$h_M(l,m,n) = -\sqrt{4\left(l^2+m^2+n^2\right)-\sigma^2}, \quad \text{if } n < 0. \tag{30}$$

Letting the initial conditions be $\phi^0 = 0$, $\psi^0 = \Psi_{\underline{n}k}$ the solution to (19)–(22) for a weakly dissipative mode becomes

$$\psi(\underline{x},t) = \left[\cos\frac{h_M(\underline{n})t}{2} + \frac{\sigma}{h_M(\underline{n})}\sin\frac{h_M(\underline{n})t}{2}\right]e^{\frac{\sigma}{2}t}\Psi_{\underline{n}k}(\underline{x}). \tag{31}$$

While the initial conditions $\phi^0 = \Phi_{\underline{n}k}$, $\psi^0 = 0$ result in

$$\psi(\underline{x},t) = \frac{2\alpha_{\underline{n}}}{h_M(\underline{n})}\sin\frac{h_M(\underline{n})t}{2}e^{\frac{\sigma}{2}t}\Psi_{\underline{n}k}(\underline{x}). \tag{32}$$

Substitution of (31) and (32) into the right-hand side of (25) yields the following moment problem: Find the surface current $J(\underline{x},t)$ together with minimum time T required for the expressions (33), (34), given below, to hold for every multi-index $\underline{n}, k$ corresponding to a weakly dissipative mode:

$$E_{\underline{n}k}^0 = \int_0^T\int_\Gamma J(\underline{x},t)\cdot\left[\cos\frac{h_M(\underline{n})t}{2} + \frac{\sigma}{h_M(\underline{n})}\sin\frac{h_M(\underline{n})t}{2}\right]e^{\frac{\sigma}{2}t}\Psi_{\underline{n}k}(\underline{x})\,d\Gamma\,dt, \tag{33}$$

$$H_{\underline{n}k}^0 = \int_0^T\int_\Gamma J(\underline{x},t)\cdot\frac{2\alpha_{\underline{n}}}{h_M(\underline{n})}\sin\frac{h_M(\underline{n})t}{2}e^{\frac{\sigma}{2}t}\Psi_{\underline{n}k}(\underline{x})\,d\Gamma\,dt. \tag{34}$$

We next follow the ideas in [10,13] to transform the expressions (33), (34) to:

$$\frac{h_M(\underline{n})H_{\underline{n}k}^0}{2\alpha_{\underline{n}}} = \int_0^T\int_\Gamma J(\underline{x},t)\cdot\sin\frac{h_M(\underline{n})t}{2}e^{\frac{\sigma}{2}t}\Psi_{\underline{n}k}(\underline{x})\,d\Gamma\,dt, \tag{35}$$

$$E_{\underline{n}k}^0 - \frac{\sigma H_{\underline{n}k}^0}{2\alpha_{\underline{n}}} = \int_0^T\int_\Gamma J(\underline{x},t)\cdot\cos\frac{h_M(\underline{n})t}{2}e^{\frac{\sigma}{2}t}\Psi_{\underline{n}k}(\underline{x})\,d\Gamma\,dt. \tag{36}$$

Thus, for every $\underline{n}$ and k, unless $n = 0$ and $l * m > 0$, substitution of (27) into expressions (35), (36) and using the orthogonality between $\bar{\Psi}$ and Ψ results in

$$\frac{h_M(\underline{n}) H^0_{\underline{n}k}}{2\alpha_{\underline{n}}} = K(\underline{n}, k) \int_0^T \sin \frac{h_M(\underline{n})t}{2} \gamma_{lmk}(t)\, dt, \tag{37}$$

$$E^0_{\underline{n}k} - \frac{\sigma H^0_{\underline{n}k}}{2\alpha_{\underline{n}}} = K(\underline{n}, k) \int_0^T \cos \frac{h_M(\underline{n})t}{2} \gamma_{lmk}(t)\, dt. \tag{38}$$

Otherwise, as discussed in Remark 3 of Section 3.4, if the index $n = 0$ and both $l > 0$, $m > 0$:

$$\frac{h_M(lm0) H^0_{lm00}}{2\alpha_{lm0}} = \int_0^T \sin \frac{h_M(lm0)t}{2} \bigl[\gamma_{lm1}(t) K(l, m, 0, 1) + \gamma_{lm2}(t) K(l, m, 0, 2)\bigr]\, dt, \tag{39}$$

$$E^0_{lm00} - \frac{\sigma H^0_{lm00}}{2\alpha_{\underline{l}m0}} = \int_0^T \cos \frac{h_M(lm0)t}{2} \bigl[\gamma_{lm1}(t) K(l, m, 0, 1) + \gamma_{lm2}(t) K(l, m, 0, 2)\bigr]\, dt. \tag{40}$$

Combining Eqs. (37) and (38) yields that unless $n = 0$ and $l * m > 0$

$$K(\underline{n}, k) \int_0^T e^{\frac{ih_M(\underline{n})t}{2}} \gamma_{lmk}(t)\, dt = E^0_{\underline{n}k} - \frac{\sigma H^0_{\underline{n}k}}{2\alpha_{\underline{n}}} + \frac{ih_M(\underline{n}) H^0_{\underline{n}k}}{2\alpha_{\underline{n}}} \stackrel{\text{def}}{=} a_{\underline{n}k}, \tag{41}$$

$$K(\underline{n}, k) \int_0^T e^{\frac{-ih_M(\underline{n})t}{2}} \gamma_{lmk}(t)\, dt = E^0_{\underline{n}k} - \frac{\sigma H^0_{\underline{n}k}}{2\alpha_{\underline{n}}} - \frac{ih_M(\underline{n}) H^0_{\underline{n}k}}{2\alpha_{\underline{n}}} \stackrel{\text{def}}{=} b_{\underline{n}k}. \tag{42}$$

While combining (39) and (40) yields, if $n = 0$, $l > 0$, $m > 0$

$$\int_0^T e^{\frac{ih_M(lm0)t}{2}} \bigl[\gamma_{lm1}(t) K(l, m, 0, 1) + \gamma_{lm2}(t) K(l, m, 0, 2)\bigr]\, dt = E^0_{lm00} - \frac{\sigma H^0_{lm00}}{2\alpha_{\underline{l}m0}} + \frac{ih_M(lm0) H^0_{lm00}}{2\alpha_{lm0}} \stackrel{\text{def}}{=} a_{lm00}, \tag{43}$$

$$\int_0^T e^{\frac{-ih_M(lm0)t}{2}} \bigl[\gamma_{lm1}(t) K(l, m, 0, 1) + \gamma_{lm2}(t) K(l, m, 0, 2)\bigr]\, dt = E^0_{lm00} - \frac{\sigma H^0_{lm00}}{2\alpha_{\underline{l}m0}} - \frac{ih_M(lm0) H^0_{lm00}}{2\alpha_{lm0}} \stackrel{\text{def}}{=} b_{lm00}. \tag{44}$$

Thus, we have obtained the moment problem (41)–(44) for the low dissipation modes.

Remark 6. Coefficients $a_{\underline{n}k}$ and $b_{\underline{n}k}$ are complex conjugates of each other.

5.2. *Moment problem for strong dissipation*

We will only have to make the minor adjustments to the solution for the moment problem in the case of weak dissipation. The differences arise only for (at most) finitely many of those modes where

$$\sigma^2 > 4\big(l^2 + m^2 + n^2\big). \tag{45}$$

Letting $\operatorname{sgn}(n)$ stand for sign of n, define the quantity

$$\tilde{h}_M(l,m,n) = \operatorname{sgn}(n)\sqrt{\sigma^2 - 4\big(l^2 + m^2 + n^2\big)}. \tag{46}$$

Calculations similar to those outlined in the previous section result in the following moment problem for high dissipation modes:

$$K(\underline{n},k)\int\limits_0^T e^{-\frac{\tilde{h}_M(\underline{n})t}{2}}\gamma_{lmk}(t)\,dt = E^0_{\underline{n}k} - \frac{\tilde{h}_M(\underline{n}) + \sigma}{2\alpha_{\underline{n}}}H^0_{\underline{n}k} \stackrel{\text{def}}{=} \tilde{a}_{\underline{n}k}, \tag{47}$$

$$K(\underline{n},k)\int\limits_0^T \frac{\sigma}{\tilde{h}_M(\underline{n})}e^{\frac{\tilde{h}_M(\underline{n})t}{2}}\gamma_{lmk}(t)\,dt = E^0_{\underline{n}k} - \frac{\tilde{h}_M(\underline{n}) - \sigma}{2\alpha_{\underline{n}}}H^0_{\underline{n}k} \stackrel{\text{def}}{=} \tilde{b}_{\underline{n}k}, \tag{48}$$

unless $n = 0$ and $l * m > 0$.

Retaining the definitions for $\tilde{a}_{lm00}$ and $\tilde{b}_{lm00}$ from (47) and (48) these equations once again take on a slightly different form when the index $n = 0$:

$$\int\limits_0^T \big[K(l,m,0,1)\gamma_{lm1}(t) + K(l,m,0,2)\gamma_{lm2}(t)\big]e^{-\frac{\tilde{h}_M(l,m,0)t}{2}}\,dt = \tilde{a}_{lm00}, \tag{49}$$

$$\frac{\sigma}{\tilde{h}_M(l,m,0)}\int\limits_0^T \big[K(l,m,0,1)\gamma_{lm1}(t) + K(l,m,0,2)\gamma_{lm2}(t)\big]e^{\frac{\tilde{h}_M(l,m,0)t}{2}}\,dt = \tilde{b}_{lm00}. \tag{50}$$

Thus if there are any eigenmodes where (45) holds, Eqs. (47)–(50) must replace the corresponding equations in the moment problem (41)–(44).

5.3. *Moment problem for the intermediate case*

Suppose that for some combination of l, m, n we have

$$\sigma^2 = 4\big(l^2 + m^2 + n^2\big). \tag{51}$$

Calculations analogous to those in Section 5.1 produce the following pair of equations:

$$E^0_{\underline{n}k} = \int\limits_0^T\int\limits_\Gamma J(\underline{x},t)\cdot\big[e^{\frac{\sigma}{2}t} + te^{\frac{\sigma}{2}t}\big]\Psi_{\underline{n}k}(\underline{x})\,d\Gamma\,dt, \tag{52}$$

$$H^0_{\underline{n}k} = \int\limits_0^T\int\limits_\Gamma J(\underline{x},t)\cdot\sigma te^{\frac{\sigma}{2}t}\Psi_{\underline{n}k}(\underline{x})\,d\Gamma\,dt. \tag{53}$$

Elimination of the $te^{\frac{\sigma}{2}t}$ term in (52)–(53) and substitution of (27) yields

$$E^0_{\underline{n}k} - \frac{1}{\sigma} H^0_{\underline{n}k} = K(\underline{n}, k) \int_0^T \gamma_{lmk}(t)\, dt, \quad \text{unless } n = 0 \text{ and } l * m > 0, \tag{54}$$

$$\frac{1}{\sigma} H^0_{\underline{n}k} = K(\underline{n}, k) \int_0^T t\gamma_{lmk}(t)\, dt, \quad \text{unless } n = 0 \text{ and } l * m > 0. \tag{55}$$

While for the combination of indices where (14) holds

$$E^0_{lm00} - \frac{1}{\sigma} H^0_{lm00} = \int_0^T \big[K(l, m, 0, 1)\gamma_{lm1}(t) + K(l, m, 0, 2)\gamma_{lm2}(t)\big]\, dt, \tag{56}$$

$$\frac{1}{\sigma} H^0_{lm00} = \int_0^T t\big[K(l, m, 0, 1)\gamma_{lm1}(t) + K(l, m, 0, 2)\gamma_{lm2}(t)\big]\, dt. \tag{57}$$

Therefore if there exist (at most finitely many) modes for which (51) holds, then each of the equations of the type (54) or (56) will replace two of the corresponding equations of either weakly or strongly dissipative kind.

6. Describing and solving for the controls in terms of bases biorthogonal to the exponentials

We will now focus on obtaining a *formal* solution to the moment problem which was defined in the previous section. Therefore, until we come to Section 7, let us assume without proof that every biorthogonal sequence that needs to exist—does. Likewise we assume that every summation involving these sequences converges when such convergence is required. From now on we set $T = 2P$, $P > \pi$.

Also we set the following notation: all sequences which are expressed in terms of multiple indices will have the indices with the subscript $_0$ held fixed while "running" over the index without the subscript (e.g., $\{h(l_0, m_0, n)\}$ means that the sequence is fixed at l_0 and m_0 while running over values of n).

6.1. Case of the weak dissipation

Starting with a sequence,

$$\big\{e^{\pm\frac{ih_M(l_0,m_0,n)t}{2}}\big\}_{n=0}^{\infty} \tag{58}$$

we define a biorthogonal sequence to (58) in $L^2(0, T)$ which we can represent as

$$\big\{\chi^{(1)}_{l_0m_0n}(t), \chi^{(2)}_{l_0m_0n}(t)\big\}_{n=0,1,2,\ldots}. \tag{59}$$

(We discuss the conditions which guarantee the existence of such a sequence in the next section.) We will use the following notation to designate the elements whose mutual inner products are nonzero:

$$\big(\chi^{(1)}_{l_0m_0n}(t), e^{\frac{ih_M(l_0,m_0,n)t}{2}}\big)_{L^2(0,T)} = \big(\chi^{(2)}_{l_0m_0n}(t), e^{-\frac{ih_M(l_0,m_0,n)t}{2}}\big)_{L^2(0,T)} = 1.$$

In order to separate the two terms of the summation inside the integral in (43)–(44) we will make certain assumptions regarding the form $\gamma_{lmk}(t)$ must take, all which will be verified a posteriori. For fixed l_0, m_0, $k = 1$, we will represent

$$\gamma_{l_0m_01}(t) = \sum_{n=0}^{\infty}\left[\xi^{(1)}_{l_0m_0n1}\chi^{(1)}_{l_0m_0n}(t) + \xi^{(2)}_{l_0m_0n1}\chi^{(2)}_{l_0m_0n}(t)\right], \quad l_0 > 0,\ m_0 > 0. \tag{60}$$

In light of Remark 3 we must reduce the number of unknowns in the moment problem by setting

$$\xi^{(1)}_{l_0m_002} = \xi^{(2)}_{l_0m_002} = 0 \tag{61}$$

in order to prevent the (infinite) system of equations corresponding to the moment problem from becoming underdetermined. Thus, for $k = 2$ the terms corresponding to $n = 0$ no longer appear:

$$\gamma_{l_0m_02}(t) = \sum_{n=1}^{\infty}\left[\xi^{(1)}_{l_0m_0n2}\chi^{(1)}_{l_0m_0n}(t) + \xi^{(2)}_{l_0m_0n2}\chi^{(2)}_{l_0m_0n}(t)\right], \quad l_0 > 0,\ m_0 > 0. \tag{62}$$

Finally, in order to avoid driving the nonexistent Fourier modes to zero, we require that functions corresponding to either $l_0 = 0$ or $m_0 = 0$ have expansions similar to (62):

$$\gamma_{l_000}(t) = \sum_{n=1}^{\infty}\left[\xi^{(1)}_{l_00n0}\chi^{(1)}_{l_00n}(t) + \xi^{(2)}_{l_00n0}\chi^{(2)}_{l_00n}(t)\right], \quad l_0 > 0, \tag{63}$$

$$\gamma_{0m_00}(t) = \sum_{n=1}^{\infty}\left[\xi^{(1)}_{0m_0n0}\chi^{(1)}_{0m_0n}(t) + \xi^{(2)}_{0m_0n0}\chi^{(2)}_{0m_0n}(t)\right], \quad m_0 > 0. \tag{64}$$

6.1.1. Solving the moment problem

We are now in a position to solve for the coefficients ξ in the expansions (60)–(64) which in turn form the components of $J(\underline{x}, t)$ in (27). Substituting (60) into (41) and (42) and utilizing the $L^2(0, T)$ biorthogonality between sequences (58) and (59) yields:

$$\xi^{(2)}_{l_0m_0n_0k_0} = \frac{a_{\underline{n}_0k_0}}{K(\underline{n}_0, k_0)}, \quad \xi^{(1)}_{l_0m_0n_0k_0} = \frac{b_{\underline{n}_0k_0}}{K(\underline{n}_0, k_0)}, \quad \text{unless } n_0 = 0 \text{ and } l_0 * m_0 > 0. \tag{65}$$

Likewise, substituting (62)–(64) into (43) and (44), using biorthogonality between (58) and (59) and using (61), gives for the remaining combination of indices:

$$\xi^{(2)}_{l_0m_001} = \frac{a_{l_0m_000}}{K(l_0, m_0, 0, 1)}, \qquad \xi^{(1)}_{l_0m_001} = \frac{b_{l_0m_000}}{K(l_0, m_0, 0, 1)}. \tag{66}$$

Note that we do not yet know anything about which functions spaces the controls, given by (27), (65), and (66), belong to.

6.2. Case of the strong dissipation

Following the familiar pattern we now turn our attention to the cases when there are modes for which the expression (45) holds. Without loss of generality, we assume for now that there do not exist any modes for which the degenerate case (51) holds. Then the moment problems (47)–(50) suggest that instead of the sequence (58), we need to use the sequence biorthogonal (in $L^2(0, T)$) to:

$$\left\{e^{\frac{\pm\tilde{h}_M(l_0,m_0,n)t}{2}}\right\}_{0\leqslant n<N} \cup \left\{e^{\frac{\pm i h_M(l_0,m_0,n)t}{2}}\right\}_{n\geqslant N}. \tag{67}$$

Therefore we now redefine $\{\chi^{(1)}_{l_0m_0n}, \chi^{(2)}_{l_0m_0n}\}_{n=0,1,2,\dots}$ to be the sequence which is biorthogonal to (67). Using it in the expansion (60)–(64), calculations analogous to those in Section 6.1 result in:

$$\xi^{(1)}_{l_0m_0nk_0} = \frac{\tilde{a}_{\underline{n}_0k_0}}{K(\underline{n}_0, k_0)}, \quad \xi^{(2)}_{l_0m_0nk_0} = \frac{\tilde{b}_{\underline{n}_0k_0}\tilde{h}_M(\underline{n}_0)}{K(\underline{n}_0, k_0)\sigma}, \quad \text{unless } n_0 = 0 \text{ and } l_0 * m_0 > 0, \tag{68}$$

$$\xi^{(1)}_{l_0m_001} = \frac{\tilde{b}_{l_0m_000}\tilde{h}_M(l_0, m_0, 0)}{\sigma K(l_0, m_0, 0, 1)}, \qquad \xi^{(2)}_{l_0m_001} = \frac{\tilde{a}_{l_0m_000}}{K(l_0, m_0, 0, 1)}. \tag{69}$$

Thus the presence of highly dissipative modes means that in the formal solution to the control problem we must replace finitely many coefficients given by (65), (66) with those given by (68), (69).

6.3. Case of the intermediate dissipation

Suppose that for some combination of $l, m, n = N \geqslant 0$ the equality (51) holds. Then, in light of (54)–(57), we wish to consider a sequence biorthogonal to

$$\{1, t\} \cup \left\{e^{\frac{\pm\tilde{h}_M(l_0, m_0, n)t}{2}}\right\}_{0 \leqslant n < N} \cup \left\{e^{\frac{\pm i h_M(l_0, m_0, n)t}{2}}\right\}_{n > N}. \tag{70}$$

Remark 7. When $N = 0$, the second set in the union (70) will be empty.

Aside from these technical differences, we solve for the appropriate coefficients in a manner which is analogous to Sections 6.1 and 6.2 (see [16, Section 5.4.3]).

7. Utilizing Riesz bases and their properties

Having derived the formal solution to the control problem in Section 6, we take up the existence of sequences biorthogonal to each of the sequences (58), (67), (70), and whether the expansions making up (60)–(64) converge in $L^2(0, T)$. In Section 9, biorthogonal sequences will be constructed explicitly for the low dissipation case. We will use various results from [26] and [1] to show that the sequences (58), (67), and (70) with one element removed from each, will form a Riesz basis in the space $L^2(0, 2\pi)$ implying that all three sequences (without any missing elements) are incomplete in $L^2(0, T)$. Existence of biorthogonal sequences follows directly from this [26].

7.1. Establishing a Riesz basis property on $L^2(0, 2\pi)$

Once again, let us fix l_0, m_0. We begin by establishing this property for the sequence (58) with the element corresponding to $n =$ "-0," $e^{-\frac{ih_M(l_0, m_0, 0)t}{2}}$, removed. Such sequence may be denoted by $\{e^{\frac{ih_M(l_0, m_0, n)t}{2}}\}_{n \in \mathbb{Z}}$. Following [26, p. 550] define the *density*, D, and *asymptotic gap*, G.

$$D \stackrel{\text{def}}{=} \lim_{n\to\infty} \frac{n}{h_M(l_0, m_0, n)/2} = \lim_{n\to\infty} \frac{n}{\sqrt{l_0^2 + m_0^2 + n^2 - (\frac{\sigma}{2})^2}} = 1,$$

$$G \stackrel{\text{def}}{=} \lim_{n\to\infty}\inf \big(h_M(l_0, m_0, n+1)/2 - h_M(l_0, m_0, n)/2\big) = 1. \tag{71}$$

There exists a positive integer N and a real number $\epsilon > 0$ such that for all n for which $|n| > N$

$$\sup_{\{n\in\mathbb{Z}:\ |n|>N\}} \left| \frac{h_M(l_0, m_0, n)}{2} - \frac{n}{D} \right| < \frac{1}{4D} - \epsilon.$$

Therefore as a consequence of [26, p. 555] when

$$T = 2\pi D = 2\pi$$

the following result holds:

Lemma 1. *Sequence* $\{e^{\frac{ih_M(l_0,m_0,n)t}{2}}\}_{n\in\mathbb{Z}}$ *forms a Riesz basis in* $L^2(0, 2\pi)$.

Next, we recognize that the sequence (67), which comes up in the strong dissipation case, is obtained by replacing finitely many elements of (58) with those of the type $e^{\frac{\pm\tilde{h}_M(l_0,m_0,n)t}{2}}$. Likewise the sequence (70) is obtained by replacing finitely many elements of (58). Thus the sequences which come up in stronger dissipation satisfy the condition of [1, Theorem II.4.11, p. 105] which says that such substitutions will preserve the minimality and the basis property of the sequence of functions. Consequently,

Lemma 2. *Sequence* (67) *with one element removed as well as sequence* (70) *with one element remowed forms a Riesz basis in* $L^2(0, 2\pi)$.

7.2. Case of $T > 2\pi$

When $T > 2\pi$, the addition of finitely many elements to the sequences which form Riesz bases in $L^2(0, 2\pi)$ will result in the incomplete sequences in $L^2(0, T)$. Consequently, the sequences (58), (67), and (70) (without any of their elements being removed) will form a Riesz basis in the $L^2(0, T)$ closure of their span, implying the existence of the biorthogonal sequences as well as the convergence of summations like (60) and (62)–(64) when some additional requirements on coefficients $\xi^{(i)}_{lmnk}$, $i = 1, 2$, are met (requirements for this convergence will be discussed in the next subsection). Defining

$$\mathcal{S} \stackrel{\text{def}}{=} \overline{\operatorname{Span}\{e^{\pm\frac{ih_M(l_0,m_0,n)t}{2}}\}}_{n=0}^{\infty}$$

with the closure taken in $L^2(0, T)$ for any $T > 2\pi$. We know (e.g., [9]) that a biorthogonal sequence $\{\chi^{(1)}_{lmn}, \chi^{(2)}_{lmn}\}$ is also a Riesz basis of $\mathcal{S}$. We have established

Lemma 3. *Sequence* (58) *as well as a sequence biorthogonal to* (58) *forms a Riesz basis in* $\mathcal{S}$.

Again since sequences (67) and (70) differ from (58) by finitely many elements, an equivalent statement to Lemma 3 holds for all of them.

Remark 8. In the case of the system (70) more care is required. It is shown in [26, pp. 551–552] that the addition of the function t regarded as an element of $L^2(0, T)$ does not destroy the minimality property of the rest of the set of complex exponentials.

Remark 9. Lemma 3 can also be easily established as a special case of Theorem 3 from [11], given the value of the asymptotic gap calculated in (71). In [11] the constants appearing in generalized Ingham inequality were estimated for the case arising in control along 1 side of a square of 2-dimensional wave equation.

7.3. Proof of Theorem 1: Establishing spectral controllability

In this section we will show that for any $T > 2\pi$, any initial data which is a linear combination of finitely many eigenfunctions of the Maxwell operator, can be driven to a zero state by the boundary controls in $L^2(0, T; L^2(\Gamma)^3)$. This is called spectral controllability [21]. In order to establish this property, we will attempt to calculate the $L^2(0, T; L^2(\Gamma)^3)$ norm of the controls, derived in Section 6. These calculations will show that in the absence of some additional requirements on the smoothness of the initial data established in Section 9, the Fourier expansion of the initial data must consist of finitely many nonzero coefficients in order for the $L^2(0, T; L^2(\Gamma)^3)$ norm of the controls to be finite. Before proceeding, we need to remark that because there are at most finitely many coefficients corresponding to strong or intermediate dissipation, then in order to be able to bound the norm of the controls, we only need to consider the weakly dissipative case.

Utilizing expansion (27), the fact that $\sigma \geqslant 0$ and the orthonormality of $\{\bar{\Psi}_{lmk}\}$ results in:

$$\begin{aligned}
&\|J(\underline{x},t)\|^2_{L^2(0,T;L^2(\Gamma)^3)}\\
&\quad\leqslant \int_0^T\int_\Gamma\Bigg|\Bigg[\sum_{m=1}^\infty \gamma_{0m0}(t)\bar{\Psi}_{0m0}(\underline{x})+\sum_{l=1}^\infty \gamma_{l00}(t)\bar{\Psi}_{l00}(\underline{x})\\
&\qquad+\sum_{l=1}^\infty\sum_{m=1}^\infty\sum_{k=1,2}\gamma_{lmk}(t)\bar{\Psi}_{lmk}(\underline{x})\Bigg]\Bigg|^2\,d\Gamma\,dt\\
&\quad=\int_0^T\Bigg[\sum_{m=1}^\infty\big|\gamma_{0m0}(t)\big|^2+\sum_{l=1}^\infty\big|\gamma_{l00}(t)\big|^2+\sum_{l,m=(1,1)}^\infty\sum_{k=1,2}\big|\gamma_{lmk}(t)\big|^2\Bigg]\,dt. \qquad (72)
\end{aligned}$$

Substituting expansions (60)–(64) into the right-hand side of (72) results in:

$$\begin{aligned}
&\|J(\underline{x},t)\|^2_{L^2(0,T;L^2(\Gamma)^3)}\\
&\quad\leqslant \int_0^T\sum_{m=1}^\infty\Bigg|\sum_{n=1}^\infty\big[\xi^{(1)}_{0mn0}\chi^{(1)}_{0mn}(t)+\xi^{(2)}_{0mn0}\chi^{(2)}_{0mn}(t)\big]\Bigg|^2\,dt\\
&\qquad+\int_0^T\sum_{l=1}^\infty\Bigg|\sum_{n=1}^\infty\big[\xi^{(1)}_{l0n0}\chi^{(1)}_{l0n}(t)+\xi^{(2)}_{l0n0}\chi^{(2)}_{l0n}(t)\big]\Bigg|^2\,dt\\
&\qquad+\int_0^T\sum_{l=1}^\infty\sum_{m=1}^\infty\Bigg|\sum_{n=0}^\infty\sum_{k=1,2}\big[\xi^{(1)}_{lmnk}\chi^{(1)}_{lmn}(t)+\xi^{(2)}_{lmnk}\chi^{(2)}_{lmn}(t)\big]\Bigg|^2\,dt. \qquad (73)
\end{aligned}$$

Due to Lemma 3 we may now apply a general theory of Riesz bases to the expression (73). Therefore (e.g., [30, Theorem 9, p. 32]), there exists a constant $B_{l_0m_0k_0} > 0$ such that when $l_0 > 0, m_0 > 0$

$$\begin{aligned}
&\left\| \sum_{n=0}^{\infty} \left(\xi^{(1)}_{l_0 m_0 n 1} \chi^{(1)}_{l_0 m_0 n}(t) + \xi^{(2)}_{l_0 m_0 n 1} \chi^{(2)}_{l_0 m_0 n}(t) \right) \right\|^2_{L^2(0,T)} \\
&\quad \leqslant B_{l_0 m_0 1} \left[\sum_{n=0}^{\infty} \left(\left|\xi^{(1)}_{l_0 m_0 n 1}\right|^2 + \left|\xi^{(2)}_{l_0 m_0 n 1}\right|^2 \right) \right].
\end{aligned} \tag{74}$$

Similar expressions, with summations beginning with $n = 1$ instead of $n = 0$, are obtained for all other combinations of k_0, l_0, m_0.

It can be seen from the solutions to the moment problem, that the right-hand side of expression of the type (74) converges. Consider for example convergence of the following summation:

$$\sum_{n=1}^{\infty} \left|\xi^{(1)}_{l_0 m_0 n k_0}\right|^2. \tag{75}$$

Substituting (65) and (42) in (75) yields

$$\begin{aligned}
&\sum_{n=1}^{\infty} \left|\xi^{(1)}_{l_0 m_0 n k_0}\right|^2 \\
&\quad = \sum_{n=1}^{\infty} \left| \frac{b_{l_0, m_0, n, k_0}}{K(l_0, m_0, n, k_0)} \right|^2 \\
&\quad = \sum_{n=1}^{\infty} \left| E^0_{l_0 m_0 n k_0} - \frac{\sigma H^0_{l_0 m_0 n k_0}}{2\alpha_{l_0 m_0 n}} - \frac{i h_M(l_0 m_0 n) H^0_{l_0 m_0 n k_0}}{2\alpha_{l_0 m_0 n}} \right|^2 \left| \frac{1}{K(l_0, m_0, n, k_0)} \right|^2 \\
&\quad \leqslant \sum_{n=1}^{\infty} \left\{ \left|E^0_{l_0 m_0 n k_0}\right|^2 + \left| \frac{\sigma H^0_{l_0 m_0 n k_0}}{2\alpha_{l_0 m_0 n}} \right|^2 + \left| \frac{i h_M(l_0 m_0 n) H^0_{l_0 m_0 n k_0}}{2\alpha_{l_0 m_0 n}} \right|^2 \right\} \left| \frac{1}{K(l_0, m_0, n, k_0)} \right|^2.
\end{aligned}$$

Each term multiplying the coefficients $E^0_{l_0 m_0 n k_0}$ or $H^0_{l_0 m_0 n k_0}$ on the right side in the above expression may be bounded by a constant which does not depend on n. As a consequence of (16) and (17), $|\frac{1}{K(l_0, m_0, n, k_0)}|^2 \leqslant C(l_0, m_0, k_0)$. Also $|\frac{i h_M(l_0 m_0 n)}{2\alpha_{l_0 m_0 n}}|^2 \leqslant \frac{1}{4}$ and $|\frac{\sigma}{2\alpha_{l_0 m_0 n}}|^2 \leqslant C(l_0, m_0)$ (different $C(l_0, m_0)$ from the one before). The summations of the squares of Fourier coefficients of the initial data will converge provided that the initial data resides in $L^2(\Omega)^3$. As mentioned in Remark 1, in this case an even stronger condition holds since the initial data resides in a subset of $H^1(\Omega)^3$. Thus, (75) converges and all the other infinite summations of this type, like right side of (74), converge by a very similar argument.

Therefore, recalling that $\xi^{(s)}_{lm02} = 0$, $s = 1, 2$, we may bound (73) by

$$\begin{aligned}
\left\| J(\underline{x}, t) \right\|^2_{L^2(0,T; L^2(\Gamma)^3)} &\leqslant \sum_{m=1}^{\infty} B_{0m0} \sum_{n=1}^{\infty} \left(\left|\xi^{(1)}_{0mn0}\right|^2 + \left|\xi^{(2)}_{0mn0}\right|^2 \right) \\
&\quad + \sum_{l=1}^{\infty} B_{l00} \sum_{n=1}^{\infty} \left(\left|\xi^{(1)}_{l0n0}\right|^2 + \left|\xi^{(2)}_{l0n0}\right|^2 \right) \\
&\quad + \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \sum_{k=1,2} B_{lmk} \left[\sum_{n=0}^{\infty} \left(\left|\xi^{(1)}_{lmnk}\right|^2 + \left|\xi^{(2)}_{lmnk}\right|^2 \right) \right].
\end{aligned} \tag{76}$$

When the indices l and m range over finitely many values, we have a common bound on the coefficients B_{lmk} in (76). Thus, if the Fourier expansion of the initial states E^0, H^0 has only finitely many (L and M) nonzero terms in the index l and m (while expansion in n may still be infinite), then (76) implies existence of a single constant $B_{\max}$ such that:

$$\begin{aligned}
&\left\| J(\underline{x}, t) \right\|^2_{L^2(0,T;L^2(\Gamma)^3)} \\
&\quad \leqslant B_{\max} \int_0^T \sum_{m=1}^{M} \left| \sum_{n=1}^{\infty} \left[\xi^{(1)}_{0mn0} \chi^{(1)}_{0mn}(t) + \xi^{(2)}_{0mn0} \chi^{(2)}_{0mn}(t) \right] \right|^2 dt \\
&\qquad + \int_0^T \sum_{l=1}^{L} \left| \sum_{n=1}^{\infty} \left[\xi^{(1)}_{l0n0} \chi^{(1)}_{l0n}(t) + \xi^{(2)}_{l0n0} \chi^{(2)}_{l0n}(t) \right] \right|^2 dt \\
&\qquad + \int_0^T \sum_{l=1}^{L} \sum_{m=1}^{M} \left| \sum_{n=1}^{\infty} \sum_{k=1,2} \left[\xi^{(1)}_{lmnk} \chi^{(1)}_{lmn}(t) + \xi^{(2)}_{lmnk} \chi^{(2)}_{lmn}(t) \right] \right|^2 dt < \infty,
\end{aligned} \tag{77}$$

where $B_{lmk} \leqslant B_{\max}$ for all admissible values of l, m, k. Therefore, we have shown our system to be spectrally controllable (in the terminology of [1], M-controllable).

Remark 10. Although calculations presented above pertain only to low dissipation, high dissipation modes will only change at most finitely many terms in the summation on the right side of (76). Therefore, the spectral controllability result will carry over to intermediate and high dissipation case.

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